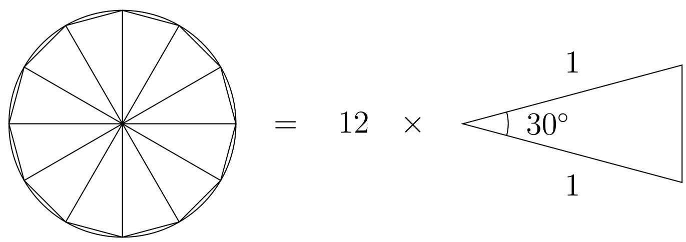
SOLUTIONS FOR ADMISSIONS TEST IN MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
WEDNESDAY 03 NOVEMBER 2021

Mark Scheme:
Each part of Question 1 is worth 4 marks which are awarded solely for the correct answer.
Each of Questions 2–7 is worth 15 marks

1

A Connect each point to the centre of the circle to split the shape into 12 isosceles triangles, each with angle $30°$ at the centre. Then the area of each triangle is $\frac{1}{2} \times 1 \times 1 \times \sin 30° = \frac{1}{4}$, and there are 12 triangles, for a total area of 3.



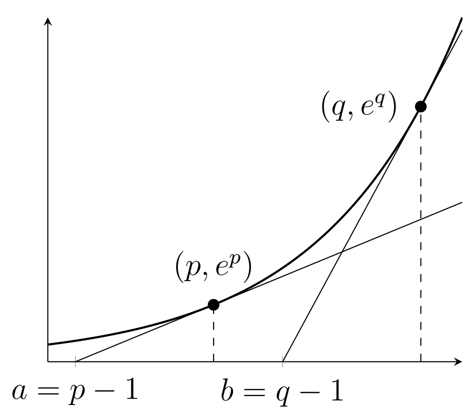
The answer is (e)

B The integral is

$$\int_0^a \sqrt{x} + x^2 \, \mathrm{d}x = \int_0^a x^{1/2} + x^2 \, \mathrm{d}x = \left[\frac{2}{3}x^{3/2} + \frac{1}{3}x^3\right]_0^a = \frac{2}{3}a^{3/2} + \frac{1}{3}a^3.$$

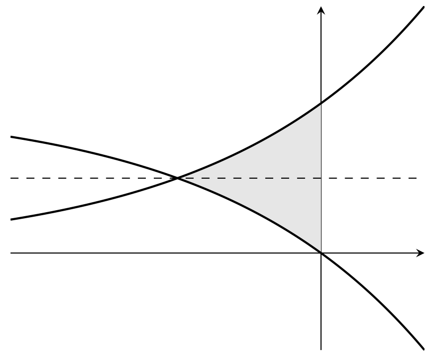
So we have $2a^{3/2} + a^3 = 15$. This factorises as $(a^{3/2} - 3)(a^{3/2} + 5) = 0$. Since $a > 0$ we want $a^{3/2} > 0$, so it's 3, so $a = 3^{2/3}$.
The answer is (c)

C The gradient at p is e^p and so the tangent is $y = e^p(x - p) + e^p$. This crosses the x-axis when $e^p(a-p)+e^p = 0$ which happens if $a = p-1$. Similarly $q = 1-b$ so $p-a = q-b$ (they're both 1).



The answer is (c)

D The intersection point is at $e^x = 1 - e^x$ which happens when $2e^x = 1$, that is $x = -\ln 2 < 0$.



The area has reflectional symmetry in the line $y = \frac{1}{2}$ so we want

$$2\int_{-\ln 2}^{0} e^x - \frac{1}{2}\,\mathrm{d}x = 2\left[e^x - \frac{x}{2}\right]_{-\ln 2}^{0} = 2\left((1) - \left(\frac{1}{2} + \frac{\ln 2}{2}\right)\right) = 1 - \ln 2$$

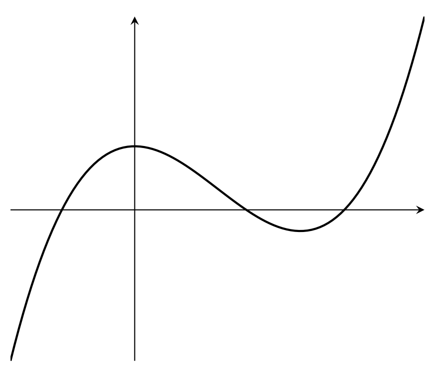
The answer is (b)

E In order to make the vector $\begin{pmatrix}10\\8\end{pmatrix}$ we would need $a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}10\\8\end{pmatrix}$ where a is the number of times we pick $\begin{pmatrix}1\\1\end{pmatrix}$ and b is the number of times we pick $\begin{pmatrix}3\\2\end{pmatrix}$.

Since we have six vectors, $a + b = 6$. Solving the simultaneous equations $a + 3b = 10$ and $a + 2b = 8$, we get $a = 4$ and $b = 2$, and we can check that $a + b = 6$ for this solution! So we want exactly two of the six vectors to be $\begin{pmatrix}3\\2\end{pmatrix}$. There are ${}^6C_2 = 15$ ways that this could happen, each with probability $\dfrac{1}{64}$, so the answer is $\dfrac{15}{64}$.

The answer is (c)

F The tangent at a is $y = (3a^2 - 3)(x - a) + (a^3 - 3a)$, which passes through $(2, 0)$ if and only if $0 = (3a^2 - 3)(2 - a) + (a^3 - 3a)$. This simplifies to $2a^3 - 6a^2 + 6 = 0$. The left-hand side is a cubic in a and we'd like to know how many roots it has.



The turning points of $2a^3 - 6a^2 + 6$ are at $a = 0$ and $a = 2$, where the value of the cubic is 6 and -2 respectively. So this cubic starts negative, rises to a positive local maximum, then decreases to a negative local minimum before rising again. There are therefore three roots for this cubic, so three values of a for which the tangent to the original cubic passes through the point $(2, 0)$.

The answer is (d)

G We can use the fact that $\sin^2(90° - n) = \cos^2(n)$ for any n, and $\sin^2 x + \cos^2 x = 1$. So we have $\sin^2 1° + \sin^2 89° = 1$ and $\sin^2 2° + \sin^2 88° = 1$ and so on up to $\sin^2 44° + \sin^2 46° = 1$. We also have $\sin^2 45° = \frac{1}{2}$ and $\sin^2 90° = 1$ for a total of $45\frac{1}{2}$.
The answer is (d)

H The function inside the brackets is $6\sin^2 x - 8\sin x + 3$ which is a quadratic for $\sin x$. We could therefore consider the quadratic $6u^2 - 8u + 3$ for $-1 \le u \le 1$. Complete the square to write this as $6\left(u - \frac{2}{3}\right)^2 + \frac{1}{3}$. This reaches a minimum value when $u = \frac{2}{3}$. For $u = \sin x$ in the range $0 \le x \le 360°$, this happens for two values of x both in $0 < x < 180°$. The value there is $\log_2\left(\frac{1}{3}\right) < 0$. Only one of the graphs reaches a negative minimum value twice in that range.
The answer is (a)

I Let's call the product of the first n terms b_n. Then we have $b_n = a_n b_{n-1}$ (that's how the product works). We also have the definition of a_n to interpret; it's one more than the previous product, so $a_n = b_{n-1} + 1$. We can use this to eliminate b_n and b_{n-1} from the previous equation, to get $a_{n+1} - 1 = a_n(a_n - 1)$. Adjust the subscripts and rearrange for $a_n = a_{n-1}(a_{n-1} - 1) + 1$.
The answer is (b)

J We must have $|AB| = |BC|$ so $\sqrt{(b-a)^2 + (c-b)^2} = \sqrt{(c-b)^2 + (d-c)^2}$.

We must also have $|BC| = |CD|$ so $\sqrt{(c-b)^2 + (d-c)^2} = \sqrt{(d-c)^2 + (a-d)^2}$.

These conditions are equivalent to $(b-a)^2 = (d-c)^2$ and $(c-b)^2 = (a-d)^2$ respectively.

Using the difference of two squares, the first is equivalent to $(a-b+c-d)(a-b-c+d) = 0$ and the second is equivalent to $(a-b+c-d)(a+b-c-d) = 0$.

In each case, we can't have both brackets equal to zero because $c \neq d$ and $b \neq c$ because the numbers are distinct. So either $a-b+c-d = 0$ or both of $a-b-c+d = 0$ and $a+b-c-d = 0$. That second case would imply that $a-c = 0$, but the numbers are distinct so that's impossible. So we're left with just the case that $a-b+c-d = 0$. We can also check that $CD = DA$ in this case, because $\sqrt{(d-c)^2 + (a-d)^2} = \sqrt{(a-d)^2 + (b-a)^2}$ rearranges to $(d-c)^2 = (b-a)^2$ which is one of the equations we already had.
The answer is (d)

(i) Setting $x = \frac{1}{2}$ in the given expression for $\ln(1-x)$ gives

$$\ln\left(\frac{1}{2}\right) = -\frac{1}{2} - \frac{(1/2)^2}{2} - \frac{(1/2)^3}{3} - \frac{(1/2)^4}{4} - \dots.$$

Then note that $\ln(1/2) = -\ln 2$ to get

$$\ln 2 = \frac{1}{2} + \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} + \frac{1}{4\times 2^4} + \dots.$$

2 marks

(ii) We have

$$\begin{aligned}\ln 2 =& \frac{1}{2} + \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} + \frac{1}{4\times 2^4} + \dots\\ <& \frac{1}{2} + \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} + \frac{1}{3\times 2^4} + \frac{1}{3\times 2^5} + \frac{1}{3\times 2^6} + \dots\end{aligned}$$

using the given inequality on each term after the first three terms. This sum is

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{3\times 2^3}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right)$$

and the sum inside the brackets is the sum of the terms of a geometric progression, so this is

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{3\times 2^3}(2) = \frac{1}{2} + \frac{1}{8} + \frac{1}{12} = \frac{17}{24}.$$

So $\ln 2 < \dfrac{17}{24}$. Also note that the terms are all positive, so

$$\ln 2 > \frac{1}{2} + \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} = \frac{16}{24}$$

We've proved that $\dfrac{16}{24} < \ln 2 < \dfrac{17}{24}$. So $k = 16$. **4 marks**

(iii) Setting $x = -\frac{1}{2}$ in the given expression for $\ln(1-x)$ gives

$$\ln\left(\frac{3}{2}\right) = \frac{1}{2} - \frac{1}{2\times 2^2} + \frac{1}{3\times 2^3} - \frac{1}{4\times 2^4} + \dots.$$

Now, using the fact that $\ln 3 = \ln(3/2) + \ln 2$, we can add the expression for $\ln 2$ found in part (i) to the expression we've just found for $\ln(3/2)$ to get

$$\ln 3 = 1 + \frac{1}{3\times 2^2} + \frac{1}{5\times 2^4} + \frac{1}{7\times 2^6} + \dots.$$

2 marks

(iv) In a similar way to part (ii), we can use the fact that $1/(7\times 2^6) < 1/(5\times 2^6)$, $1/(9\times 2^8) < 1/(5\times 2^8)$ and so on to write

$$\begin{aligned}\ln 3 <& 1+\frac{1}{3\times 2^2}+\frac{1}{5\times 2^4}+\frac{1}{5\times 2^6}+\frac{1}{5\times 2^8}+\dots\\ =&1+\frac{1}{3\times 2^2}+\frac{1}{5\times 2^4}\left(1+\frac{1}{4}+\frac{1}{4^2}+\dots\right)\\ =&1+\frac{1}{12}+\frac{1}{80}\left(\frac{4}{3}\right)\\ =&\frac{11}{10}\end{aligned}$$

so $\ln 3 < \frac{11}{10}$. Also note that the terms are all positive, so

$$\ln 3 > 1+\frac{1}{3\times 2^2}=\frac{13}{12}$$

4 marks

(v) Take logarithms base e. We're asked to compare $17\ln 3$ against $13\ln 4$ ($\ln x$ is an increasing function of x so it's sufficient to compare these).

We know that $17\ln 3 > \frac{17\times 13}{12}$ and that $13\ln 4 = 26\ln 2 < \frac{26\times 17}{24} = \frac{13\times 17}{12}$.

So putting it all together, $13\ln 4 < \frac{17\times 13}{12} < 17\ln 3$. That means that $4^{13} < 3^{17}$.

3 marks

3

Different alternative solutions are indicated with (Alt1), (Alt2), and so on.

(i) The value at $x = 0$ is 0 so $p = 0$. This is a turning point so $p'(0) = 0$.

(Alt1) The last two coefficients are zero so $p(x) = x^2q(x)$.

(Alt2) It's a repeated root, so x must be a factor at least twice. **3 marks**

(ii) $r(x) = (x - a)^2q(x)$ where $q(x)$ is a polynomial, or equivalently $r(x) = (x - a)^2q(x - a))$. If we translate the graph of this polynomial a units to the left then we get a polynomial with turning point at $(0, 0)$, like in (i). So translate that a units to the right to get an expression for this polynomial. **2 marks**

(iii) (a) There must be a factor of $(x - a)^2$ by part (ii), and similarly there must be a factor of $(x + a)^2$. The function $f(x)$ is a polynomial of degree 4, so we must have $f(x) = A(x - a)^2(x + a)^2$. The coefficient A could be any real number. **3 marks**

(b) Reflection in the y-axis. We can check that $f(-x) = f(x)$ by working out

$$f(-x) = A(-x - a)^2(-x + a)^2 = A(x + a)^2(x - a)^2 = f(x).$$

2 marks

(c) The third turning point must be at $x = 0$ because of the symmetry we found in the previous part. If it wasn't at $x = 0$ then there would be a fourth turning point symmetrically opposite the y-axis, but a degree 4 polynomial can only have three turning points.
1 mark

(iv) (Alt1) Yes, start with $A(x - 1)^2(x + 1)^2$ from part (iii), which had turning points at $(-1, 0)$ and $(1, 0)$ and $(0, A)$. Then translate one to the right and set $A = 3$ to get $3x^2(x - 2)^2$.

(Alt2) Or start with part (i) and write $p(x) = x^2(ax^2 + bx + c)$. Then use the information that the value at $x = 2$ is zero, the information that there's a turning point at $x = 1$, and the information that the value there is 3, to solve for $a = 3$, $b = -12$, $c = 12$. Check that there really is a turning point at $x = 2$. **2 marks**

(v) No. If we had such a polynomial, then we could translate it $2\frac{1}{2}$ units left and 6 units down so that it had turning points at $(\pm\frac{3}{2}, 0)$. Then part (iii) applies, but the third turning point is not at $x = 0$, it's at $x = -\frac{1}{2}$. **2 marks**

4

Different alternative solutions are indicated with (Alt1), (Alt2), and so on.

(i) (Alt1) The slice of cake is a rectangle below y plus a triangle above, with area

$$xy + \frac{(k-y)x}{2} = \frac{xk}{2} + \frac{xy}{2} = \frac{x(k+y)}{2}$$

(Alt2) Quote the area of a trapezium.

Checking, when $x = 1$ and $y = 1$ and $k = 1$, this gives $1 \times 2/2 = 1$. **3 marks**

(ii) We could instead take, for example, $x = \frac{4}{3}$ and $y = \frac{1}{2}$. Anything with $x(y+1) = 2$ works, provided that $0 \le x \le 2$ and $0 \le y \le 2$. **1 mark**

(iii) We would need $0 \le k \le 2$ for the point to lie on the side of the cake.

(Alt1) We have Area $= 1$ so $x(k+y) = 2$. Let's use the inequalities for k.

- $k \ge 0$ so $2 = x(k+y) \ge x(0+y)$ as $x \ge 0$. That's $xy \le 2$.
- $k \le 2$ so $2 = x(k+y) \le x(2+y)$. That's $2 \le x(2+y)$.

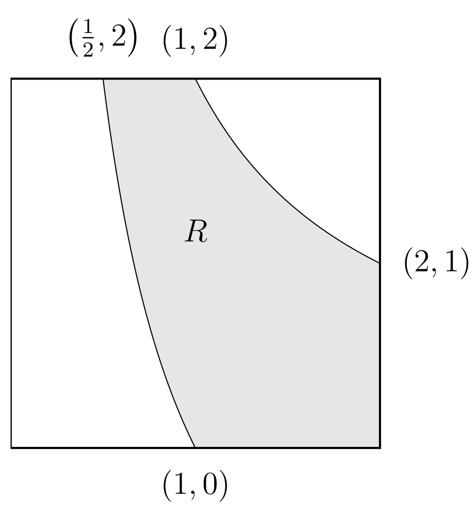
(Alt2) Rearrange Area $= 1$ for k to get $k = \frac{2}{x} - y$.

- $k \ge 0$ so $\dfrac{2}{x} - y \ge 0$ so $2 - xy \ge 0$ as $x \ge 0$.
- $k \le 2$ so $\dfrac{2}{x} - y \le 2$ so $2 - xy \le 2x$ as $x \ge 0$.

3 marks

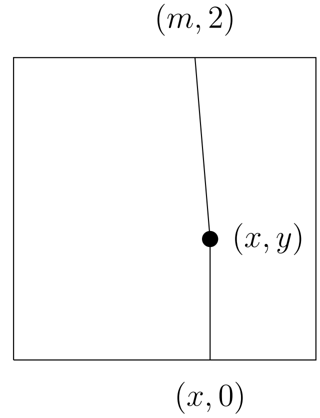
(iv) The first inequality describes a region with boundary $xy = 2$. This curve crosses $y = 2$ at $x = 1$ and crosses $x = 2$ at $y = 1$. It does not cross the other sides of the cake.

The second inequality describes a region with boundary $x(y+2) = 2$. This curve crosses $y = 2$ at $x = \frac{1}{2}$ and crosses $y = 0$ at $x = 1$. It does not cross the other sides of the cake. The region R looks like this:



3 marks

(v) In this case



Repeating the steps above for this new case, the area of the piece of cake will be

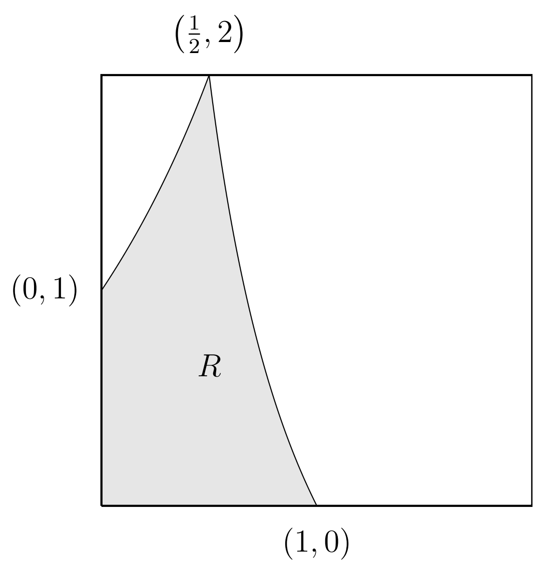
$$xy + m(2-y) + \frac{(x-m)(2-y)}{2} = xy + \frac{(2-y)(x+m)}{2}.$$

If this is 1 then $2xy + (2-y)(x+m) = 2$. Like before, we need $0 \le m \le 2$.
(Alt1)

- $2 = xy + (2-y)(x+m) \ge 2xy + (2-y)x$ so $2 \ge x(2+y)$.
- $2 = xy + (2-y)(x+m) \le 2xy + (2-y)(x+2)$ so $0 \le xy + 2x - 2y + 2$ which we could instead write as $y(2-x) \le 2(x+1)$ or even $y \le \dfrac{6}{2-x} - 2$ if $x \ne 2$.

(Alt2) Rearrange the Area $= 1$ statement for $m = \dfrac{(2-2xy)}{2-y} - x$ and use $0 \le m \le 2$ to get the same inequalities.

The region this time looks like this:



5 marks

5

(i) We have

$f(3) = 1$ The only possibility is (1, 1, 1)
$f(4) = 0$ There are no triangular triples with perimeter 4.
$f(5) = 3$ The possibilities are (1, 2, 2), (2, 1, 2), or (2, 2, 1), which count as distinct.
$f(6) = 1$ The only possibility is (2, 2, 2)

2 marks

(ii) Suppose $a \leq b \leq c$. Then we have $a+b > c$. We need to check that $(a+1)+(b+1) > (c+1)$, which is true because $(a+1)+(b+1) = a+b+2 > c+2 > c+1$. **1 mark**

(iii) Without loss of generality, say $x \leq y \leq z$. First clearly, we can't have $x = 1$, as in that case $y \leq z-1$, as $x+y+z$ is even, so y and z must have different parity. So $x, y, z \geq 2$. We need to check that $(x-1)+(y-1) > (z-1)$. Since $x+y > z$, we have $(x-1)+(y-1) \geq (z-1)$. If this holds with equality, we have $(x+y+z-2) = 2z-1$. However, the LHS is even and the RHS is odd, so this can't hold with equality.

3 marks

(iv) Given any triangular triple (a, b, c) such that $a+b+c = 2k-3$, $(a+1, b+1, c+1)$ is a triangular triple with $(a+1)+(b+1)+(c+1) = 2k$ by part (ii). Likewise, for any triple (x, y, z) with $x+y+z = 2k$, by part (iii) $(x-1, y-1, z-1)$ is a triangular triple with $(x-1)+(y-1)+(z-1) = 2k-3$. Thus these triples are in one-to-one correspondence, and so there's the same number of each $f(2k-3) = f(2k)$.

2 marks

(v) (a) We have that $a+b > c$ if and only if $a+b+c > 2c$. Since the left-hand side is $2S$, this happens if and only if $c < S$. Likewise, $a+c > b$ if and only if $b < S$ and $b+c > a$ if and only if $a < S$.

We should perhaps also check that given $a < S$ and $b < S$ and $c < S$ and $a+b+c = 2S$, then all of (a, b, c) are positive numbers. This is true because since $b < S$ and $c < S$, we have $b+c < 2S$. Since $2S = a+b+c$ this means that $a > 0$. and similarly for the others.

2 marks

(b) Note that a and b and c all have to be between 2 and $S-1$ inclusive by parts (iii) and (v)(a). We have $c = 2S-a-b$ and $c \leq S-1$, so $2S-a-b \leq S-1$, which we can rearrange for $S-a+1 \leq b$. Remember that $b \leq S-1$. So for a given value of a, we have exactly $(S-1)-(S-a) = a-1$ possible values of b, and then c is uniquely determined by $a+b+c = 2S$.

So the number of triangular triples is given by

$$f(P) = \sum_{a=2}^{S-1}(a-1) = \sum_{a=1}^{S-2} a = \frac{(S-2)(S-1)}{2}.$$

4 marks

(vi) We know that $f(21) = f(24) = \frac{11 \cdot 10}{2} = 55$.

1 mark

(i) (a) The third smallest entry must be in one of the cells $(2,1)$, $(3,1)$, $(1,2)$, or $(1,3)$.

1 mark

(b) The number in cell (i,j) is greater than or equal to all numbers in the rectangle from $(1,1)$ to (i,j). So the k^{th} smallest number can only be in cell (i,j) if $ij \le k$. Conversely, if $ij \le k$ then the k^{th} smallest number may be in cell (i,j), because the ij elements in the rectangle are smaller, and then:

- if $i > 1$ there could be precisely $k - ij$ more numbers, in the first row in columns $j+1$ onwards, which are smaller than the number at (i,j).
- if $i = 1$ there could be precisely $k - ij$ more numbers, in the first column in row 2 onwards, which are smaller than the number at (i,j).

3 marks

(ii) First check the element in the top-right cell. If it's equal to y then we're done. Otherwise, if it's bigger than y then everything in the right-most column is larger than y and can be eliminated. On the other hand, if it's less than y, then everything in the top-most row must be smaller than y and can be eliminated. Repeat this process. After $m+n-1$ inspections, we've either found y or eliminated all the rows and columns, in which case y does not appear in the table. (Other procedures work, e.g. start at the bottom-left corner). **4 marks**

(iii)

A:

33	92	46	24
25	26	37	8
49	40	81	22

$\rightarrow$ B:

24	33	46	92
8	25	26	37
22	40	49	81

$\rightarrow$ C:

8	25	26	37
22	33	46	81
24	40	49	92

2 marks

(iv) Obviously the columns are sorted because of how C is made from B.

Consider cell (i,j) of C and compare it with cell (i,k) with $k < j$, in the same row.

The first i numbers in column j of C are all smaller than or equal to the element in cell (i,j) because the column is sorted. Each of those i numbers came from table B, where it was bigger than *some* element of column k, because the rows of B were sorted.

In table C those elements are still in column k and at least one of them must be in row r_1 for some $r_1 \ge i$ (there are i of them so they can't all be in the top $i-1$ rows). Write r_2 for the row of the corresponding element in column j of C which was in the same row of B as this element.

The element (i,j) is bigger or equal to (s,j) (same column), which is bigger than (r,k) (was in the same row of B), which is bigger or equal to the cell (i,k) (same column). So we're done.

5 marks

7

Different alternative solutions are indicated with (Alt1), (Alt2), and so on.

(i) (Alt1) The function f is 1 exactly when at least one of its inputs is 1 *and* at least one of its inputs is 0.

(Alt2) The function f is 1 exactly when the maximum of the inputs is 1 and the minimum is zero.

(Alt3) The function f is 1 if and only if not all the inputs are the same. **1 mark**

(ii) (a) $\mathsf{majority}(x_1, x_2) = \mathsf{min}(x_1, x_2)$. Other expressions are possible. **1 mark**

(b) $\mathsf{majority}(x_1, x_2, x_3) = \mathsf{max}(\mathsf{min}(x_1, x_2), \mathsf{min}(x_2, x_3), \mathsf{min}(x_3, x_1))$. Other expressions are possible. **2 marks**

(iii) There are 6 possible Boolean functions of two variables that can be represented using only $\mathsf{majority}$ functions with 3 inputs. Other than $\mathsf{max}(x_1, x_2)$, they are:

(a) The constant 0 function: $\mathsf{majority}(0, 0, 0)$.

(b) The function that takes the same value as x_1: $\mathsf{majority}(x_1, x_1, 0)$ or $\mathsf{majority}(x_1, x_1, x_1)$

(c) The function that takes the same value as x_2: $\mathsf{majority}(x_2, x_2, 0)$ or $\mathsf{majority}(x_2, x_2, x_2)$

(d) The function $\mathsf{min}(x_1, x_2)$: $\mathsf{majority}(x_1, x_2, 0)$.

(e) The constant 1 function: $\mathsf{majority}(1, 1, 1)$.

Other expressions are possible for these functions. **4 marks**

(iv) (Alt1) The function xor, given by $g(0, 0) = g(1, 1) = 0$ and $g(0, 1) = g(1, 0) = 1$ cannot be represented using composition of $\mathsf{majority}$ functions.

This is because increasing an input of $\mathsf{majority}$ can only increase the output if it changes at all (and this is also true when you combine $\mathsf{majority}$ functions together). But xor doesn't obey this property, as $g(0, 1) = 1$, but $g(1, 1) = 0$.

(Alt2) There are nine others, including things like $g(x_1, x_2) = \mathsf{flip}(x_1)$ or the function with $g(0, 0) = 1$ but zero otherwise.

3 marks

(v) (a) Yes, $\mathsf{majority}(x_1, x_2, x_3, x_4) \equiv \mathsf{majority}(z_1, z_2, z_3, z_4, 1)$. Note that if at least three x_i are 1, then at least 2 z_i are 1. Likewise, if at most 2 x_i are 1, then at most 1 of the z_i is 1. **2 marks**

(b) No, because there are $x_1 = 1, x_2 = x_3 = x_4 = 0$ and $x_1 = x_2 = x_3 = x_4 = 0$ both yield $z_1 = z_2 = z_3 = z_4 = 0$. However, $\mathsf{parity}(1, 0, 0, 0) = 1$ and $\mathsf{parity}(0, 0, 0, 0) = 0$, and for any g, $g(0, 0, 0, 0)$ is either 0 or 1. **2 marks**